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Tunneling Effect of Quantum Stochastic Process

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In Nelson's theory, quantum mechanical trajectory of a particle is described in terms of a stochastic process instead of a wave function. This stochastic process enables us to trace the position of a quantum particle. We study the trajectories for the superposition of plane wave functions and for tunneling effects.

1 Introduction

In quantum mechanics, the positon and momentum of a particle can not be determined simultaneously because of the uncertainty principle. Many probabilistic approaches have been tried to see the trajectories of quantum particles. In Nelson's theory, a quantum particle is subject to a stochastic differential equation which is described with a drift velocity determined from a wave function and Wiener process. We will show in this paper how a quantum particle behaves for the superposition of plane waves and how tunneling effects can occur in this frame work.

2 Nelson's method

In Nelson's construction of a quantum stochastic process[1][2], trajectories of quantum particles are determined by the following steps:

1. Solve the Schrödinger equation (1)
2. Calculate the drift velocity (2)
3. Integrate the stochastic differential equation (3)

Schrödinger equation :

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\Psi(\mathrm{X},t) = \imath\hbar\frac{\partial}{\partial t}\Psi(\mathrm{X},t), \tag{1}$$

Drift velocity :

$$\mathrm{b}(\mathrm{X},t) = \frac{\hbar}{m}\left(R_e\frac{\nabla\Psi}{\Psi} + I_m\frac{\nabla\Psi}{\Psi}\right), \tag{2}$$

Stochastic differential equation :

$$d\mathrm{X}(t) = \mathrm{b}(\mathrm{X},t)dt + \sqrt{\frac{\hbar}{2m}}d\mathrm{W}(t), \tag{3}$$

where W is a Wiener process.

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3 Superposition of Free Wave Functions

Wave functions of free particles with momentum p and -p are

$$\Psi_1(x,t) = A\exp\left[\frac{i}{\hbar}\left(px - \frac{p^2}{2m}t\right)\right], \tag{4}$$

$$\Psi_2(x,t) = B\exp\left[\frac{i}{\hbar}\left(-px - \frac{p^2}{2m}t\right)\right], \tag{5}$$

respectively. We study the trajectories of a quantum particle under the superposition of wave functions Ψ_1 and Ψ_2,

$$\Psi_s = A\exp\left[\frac{i}{\hbar}\left(px - \frac{p^2}{2m}t\right)\right] + B\exp\left[\frac{i}{\hbar}\left(-px - \frac{p^2}{2m}t\right)\right]. \tag{6}$$

The drift velocities b of the above wave functions are given by

$$b_1 = p/m, \tag{7}$$

$$b_2 = -p/m, \tag{8}$$

$$b_s(x) = \frac{1}{m}\cdot\frac{A^2 - B^2 - 2pAB\sin(2px/\hbar)}{A^2 + B^2 + 2AB\cos(2px/\hbar)}, \tag{9}$$

respectively. Figure 1 shows the drift velocities of each state. The drift velocities b_1 and b_2 are constant and b_s shows oscilatory behaviour. Integrating the corresponding stochastic differntial equations, we have

$$X_1(t) = X_1(0) + \frac{p}{m}t + \sqrt{\frac{\hbar}{2m}}(W(t) - W(0)), \tag{10}$$

$$X_2(t) = X_2(0) + \frac{-p}{m}t + \sqrt{\frac{\hbar}{2m}}(W(t) - W(0)), \tag{11}$$

$$X_s(t) = X_s(0) + \frac{1}{m}\cdot\frac{A^2 - B^2 - 2pAB\sin(2px/\hbar)}{A^2 + B^2 + 2AB\cos(2px/\hbar)}\cdot t + \sqrt{\frac{\hbar}{2m}}(W(t) - W(0)). \tag{12}$$

Figure 2 shows trajectories of X_1, X_2 and X_s. The process X_s stays between X_1 and X_2. We chose following parameters : A=1.0, B=0.7,m=1.0, p=10.0, $\hbar = 0.1$.

Figure 3 shows the behavior of X_s at large time scale. We can see that X_s is approaching X_1 which has larger amplitude. Integrating the b_s over a period, we can see the average velocity is same with original one with larger amplitede (in this case b_1). This is the origin of the behaviour.

Suppose the wave function is superposed with equal weight A = B. In Figure 5,we can see that X_s shows non-local behaviour because of the periodic divergence of the drift velocity (cf. Figure 4). The uncertainty of the position owing to the equal weight superposition is represented as such a non-locality in the quantum stochastic process.

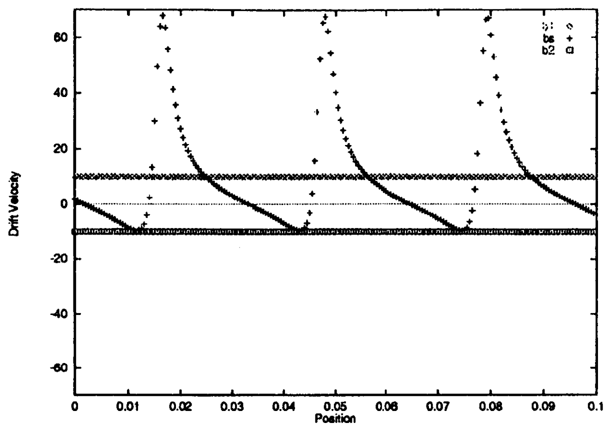

Figure 1: Position and drift velocities in A=1.0, B=0.7

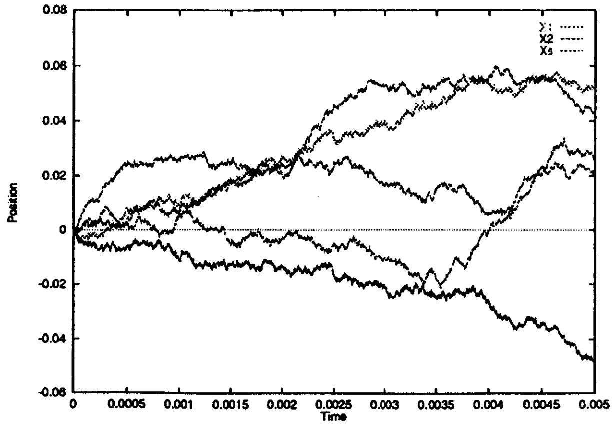

Figure 2: Quantum stochastic process X_1, X_2, X_s

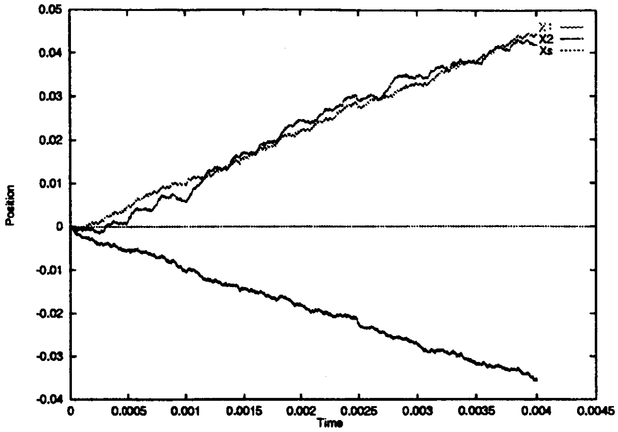

Figure 3: Quantum stochastic process X_1, X_2, X_s in A=1.0 B=0.7

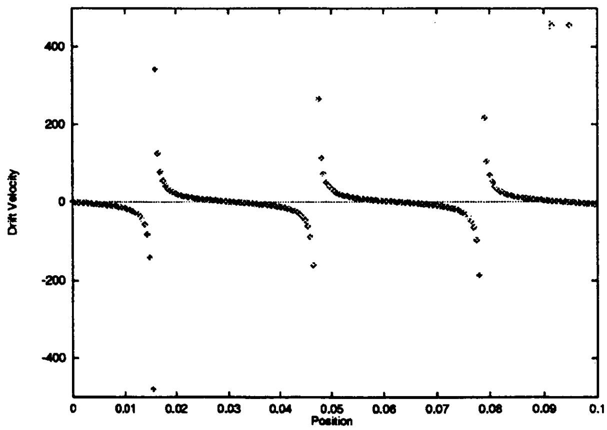

Figure 4: Position x and drift velocity $b_s$in A=B=1.0

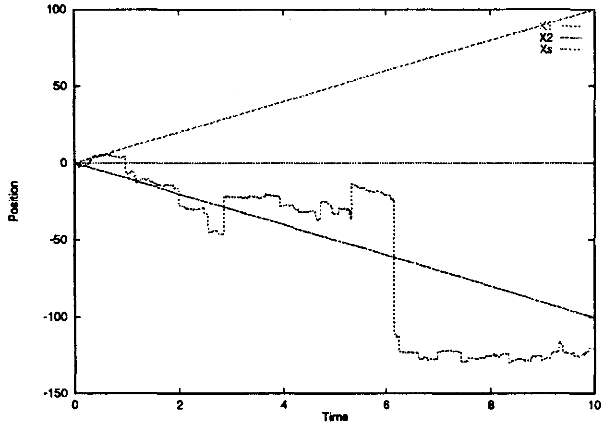

Figure 5: Quantum stochastic process X_1, X_2, X_s in A=B=1.0

4 Tunneling Effect in one dimension

We consider the quantum tunneling effect as an application of the superposition of free wave functions. For the potntial

$$V(x) = \begin{cases} 0 & x \leq 0 \\ V & 0 \leq x \leq A \\ 0 & x \geq A, \end{cases} \tag{13}$$

we have next drift velocities,

$$b1 = \frac{\hbar}{m} \frac{k^3\kappa^2 - k(\kappa^2 + k^2)\left[k\kappa \cosh\kappa a \sinh(\kappa a)\cos(2kx) + \frac{1}{2}(k^2 - \kappa^2)\sinh^2\kappa a \sin(2kx)\right]}{k^2\kappa^2\cosh^2\kappa + (k^4\cos^2 kx + \kappa^4\sin^2 kx)\sinh^2\kappa a - \kappa k(k^2 + \kappa^2)\cosh\kappa a \sinh\kappa a \sin(2kx)} \tag{14}$$

$$b2 = \frac{\hbar}{m}\sqrt{\frac{2(V-E)}{m}} \cdot \frac{V\sinh 2\sqrt{2m(V-E)}(x-a)/\hbar + 2\sqrt{2E(V-E)}}{V - 2E + V\cosh 2\sqrt{2m(V-E)}(x-a)/\hbar} \tag{15}$$

$$b3 = \sqrt{\frac{2E}{m}}, \tag{16}$$

where E is the insident energy and $k = \sqrt{2mE}/\hbar, \kappa = \sqrt{2m(V-E)}/\hbar$.

Figure 6 shows drift velocity in each region. We have set $E = 4.1, m = 1.0, \hbar = 1.0, V = 4.5, A = 0.5$.

The quantum stochastic processes are easily obtained as

$$X_1(t) = X_1(0) + b1t + \sqrt{\frac{\hbar}{2m}}(W(t) - W(0)), \tag{17}$$

$$X_2(t) = X_2(0) + b2t + \sqrt{\frac{\hbar}{2m}}(W(t) - W(0)), \tag{18}$$

$$X_3(t) = X_3(0) + b3t + \sqrt{\frac{\hbar}{2m}}(W(t) - W(0)). \tag{19}$$

When the quantum stochastic process has no fluctuation, the original stochastic differential equation becomes

$$dX(t) = bdt. \tag{20}$$

Figure 7 shows trajectories of free particles without the fluctuation for various initial positions X(0). The parameters are $m = 1.0, \hbar = 1.0, A = 1.0$. The trajectories converge to some fixed points. We can see the Wiener process plays an essential role in the quantum tunneling. Trajectories of tunneling effects are shown in Figure 8. The tunneling probability is set to 0.015 with $m = 1.0, \hbar = 1.0, X(0) = -0.5, E = 4.1$. We can see the tunneling effects are also observed and the tunneling probability shows good agreement with the expected value.

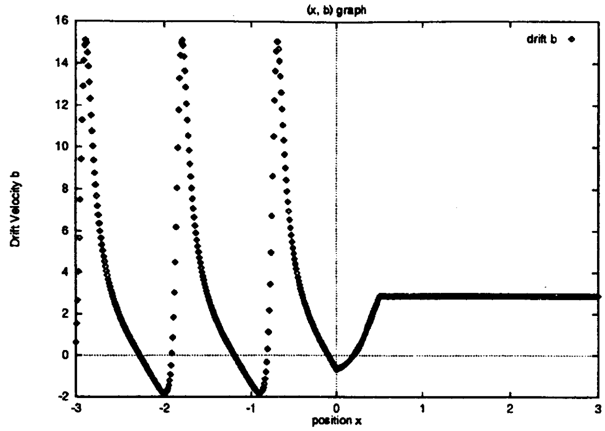

Figure 6: Position and drift velocity

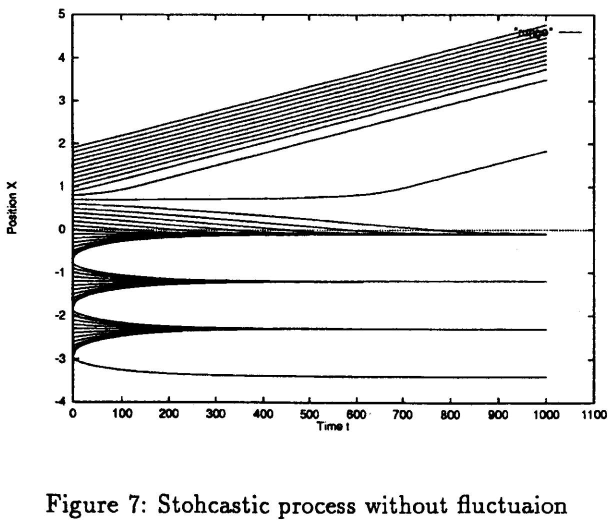

Figure 7: Stohcastic process without fluctuaion

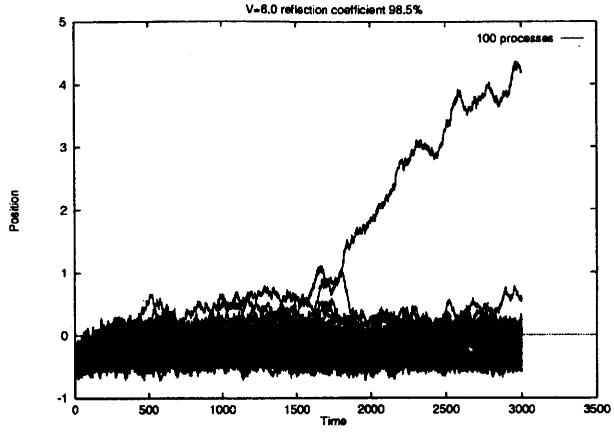

Figure 8: Quantum stochastic process in tunneling effect

5 Conclusion

We investigated the trajectories of quantum particles in the superposition of plane wave functions and tunneling effects. In the superpositon, we can see X_s approaches to the process with larger amplitude. When the amplitudes are same, X_s shows a non-local behaviour because of the divergence of the drift velocity. Tunneling effects are also observed in the quantum stochastic process. It can be seen that Wiener process is essential in the tunneling effects.

It is expected that probabilistic approaches of quantum mechanics will become an important methods to study the problems of measurements.

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